Lecture Note Of

Heat Transfer

By

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GENERAL DIFFERENTIAL EQUATION FOR HEAT CONDUCTION IN CARTESIAN COORDINATES

This is also known as heat diffusion equation or, simply heat equation. Consider a homogeneous body within which there is no bulk motion and heat transfer occurs in this body by conduction. Temperature distribution within the body at any given instant is given by: T(x, y, z, τ). The coordinate system used in this derivation is given in Fig. (1)

Consider a differential volume element dx.dy.dz from within the body as shown. It has six surfaces Further, the body is assumed to be rigid, i.e. negligible work is done on the body by external mechanical forces.

Let us make an energy balance on this differential element. Let us list out the various energy terms involved: first, there is energy conducted into the element; second, there is energy conducted out of the element; third, for generality, let there be energy generated within the element, say, due to chemical reaction or nuclear fission, etc. Net heat conducted into the element in conjunction with the heat generated within the element, will obviously cause an increase in the energy content (or the internal energy) of the element.

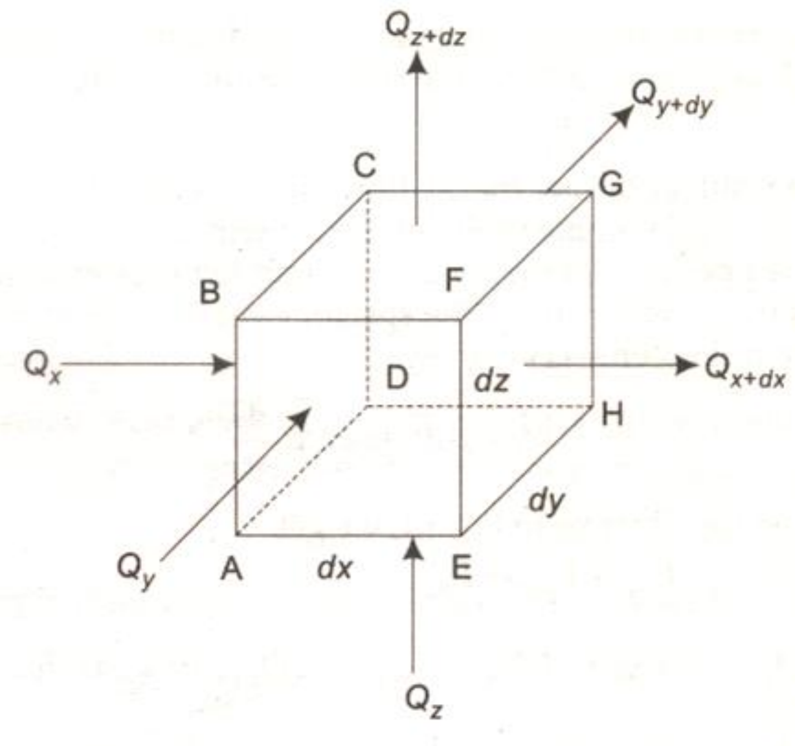


Figure (1)

We can write it mathematically as

$$E_{in} - E_{out} + E_{gen} = E_{st} \ldots\ldots\ldots.1$$

where,

E_{in} = energy entering the control volume per unit time.

E_{out} = energy leaving the control volume per unit time.

E_{gen} = energy generated within the control volume per unit time
E_{st} = energy storage within the control volume per unit time.

To calculate E_{in}. Energy enters the differential control volume from all the three sides by conduction only, since the control volume is embedded within the body considered.
Let the energy entering the control volume in the X-direction through face ABCD be Qx. Similarly, Qy and Qz enter the control volume from the faces ABFE and DAEH as shown in the Fig. (1).
$E_{in} = Qx + Qy + Qz$ … … … … ….2
To calculate E_{out}. Energy entering the control volume in the X-direction at face ABCD leaves the control volume at the opposite face EFGH. This is designated as Q_{x+dx} Similarly, Q_{y+dy} and Q_{z+dz} leave the control volume from the surfaces opposite to the ones at which they entered. Therefore, we write,

$$E_{out} = Q_{x+dx} + Q_{y+dy} + Q_{z+dz} \dots\dots\dots\dots 3$$

Now, from calculus, we know that Q_{x+dx} etc. can be expressed by a Taylor series expansion, where, neglecting the higher order terms, we can write,

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} . dx \dots\dots\dots\dots\dots 4.1$$

$$Q_{y+dy} = Q_y + \frac{\partial Q_y}{\partial y} . dy \dots\dots\dots\dots\dots 4.2$$

$$Q_{z+dz} = Q_z + \frac{\partial Q_z}{\partial z} . dz \dots\dots\dots\dots\dots 4.3$$

To calculate E_{gen}. Let there be uniform heat generation within the volume at a rate of q_g (W/m3). Heat generation is a volume phenomenon, i.e. heat is generated throughout the bulk of the body so, note its units (W/m^3). As mentioned earlier, heat may be generated within the body due to passage of an electric current, a chemical reaction, nuclear fission, etc. Then, for the differential control volume $dx.dy.dz$, we can write,

$$E_{gen} = q_g dx.dy.dz \dots\dots\dots\dots 5$$

to calculate E$_{st}$. As a result of the net energy flow into the control volume from all the three directions and the heat generated within the control volume itself, internal energy of the control volume increases. This will manifest itself as an increase in the temperature of the control volume. Let the temperature of the control volume increase by dT in time $d\tau$. Then, if ρ is the density and c_p, the specific heat of the material of the control volume, rate of increase of internal energy of control volume is given by:

$$E_{st} = \rho.dx\, dy\, dz.c_p \frac{\partial T}{\partial \tau} \dots\dots\dots\dots 6$$

Now, substituting for all terms in Eq. 1, we get,

$$E_{in} - E_{out} + E_{gen} = E_{st}$$

$$(Q_x + Q_y + Q_z) - (Q_{x+dx} + Q_{y+dy} + Q_{z+dz}) + q_g dx.dy.dz = \rho.dx\, dy\, dz.c_p \frac{\partial T}{\partial \tau}$$

$$-\left(\frac{\partial Q_x}{\partial x}\,.dx+\frac{\partial Q_y}{\partial y}\,.dy+\frac{\partial Q_z}{\partial z}\,.dz\right)+q_g dx.dy.dz=\rho\, c_p\frac{\partial T}{\partial \tau}\;dx\;dy\;dz\;\ldots\ldots..7$$

Now let us bring in Fourier's law of heat conduction.

$$Q_x=-kA_x\frac{\partial T}{\partial x}=-kdydz\frac{\partial T}{\partial x}\ldots\ldots\ldots 8.1$$
$$Q_y=-kA_y\frac{\partial T}{\partial y}=-kdxdz\frac{\partial T}{\partial y}\ldots\ldots\ldots 8.2$$
$$Q_z=-kA_z\frac{\partial T}{\partial z}=-kdxdy\frac{\partial T}{\partial z}\ldots\ldots\ldots 8.3$$

Substituting Eq.(8), in Eq.(7), and dividing by $dx.dy.dz$, we obtain,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+q_g=\rho\, c_p\frac{\partial T}{\partial \tau}$$
$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q_g=\rho\, c_p\frac{\partial T}{\partial \tau}$$

This is the general form of heat diffusion equation in Cartesian coordinates for time dependent (i.e. unsteady state) heat conduction, with uniform heat generation within the body. This is a very important basic equation for conduction analysis. It has to be solved with appropriate initial and boundary conditions to get the temperature distribution within the body as a function of spatial and time coordinates. Of course, the heat transfer rate is calculated applying the Fourier's law, once the temperature distribution is known.

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{q_g}{k}=\frac{\rho\, c_p}{k}\frac{\partial T}{\partial \tau}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$
$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}\ldots\ldots\ldots 9$$

Where, $\alpha=k/\rho c_p$ is thermal diffusivity.

Solution of general form of heat diffusion equation as given in Eq.9 is rather complicated. However, in many practical applications, we make simplifying assumptions and the resulting equations are easily solved. For example:

1) **Steady state**: This means that the temperature at any position does not change with time, i.e. $\frac{\partial T}{\partial \tau}=0$

   Eq.9, becomes:

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{q_g}{k}=0$$

   This is known as Poisson equation and is for steady state, three-dimensional heat conduction with heat generation, with constant thermal conductivity, in Cartesian coordinates.

2) **With no Internal heat generation**: This means that q_g term is zero. So, Eq. 9 becomes,

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

This is known as Diffusion equation, and it represents time dependent, three-dimensional heat conduction, with no internal heat generation, and with constant thermal conductivity, in Cartesian coordinates.

3) **Steady state, with no Internal heat generation**: This means that q_g and $\frac{\partial T}{\partial \tau}$ are zero. So, Eq. 9 becomes.

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=0$$

This is known as Laplace equation, and it represents steady state, three-dimensional heat conduction with no internal heat generation, with constant thermal conductivity, in Cartesian coordinates.

4) **One-dimensional, steady state, with no internal heat generation**: This means that.

$$\left(\frac{\partial^2 T}{\partial y^2}=\frac{\partial^2 T}{\partial z^2}=0\right),\ \left(q_g=0\right) and\ \left(\frac{\partial T}{\partial \tau}=0\right)$$

So Eq. 9 becomes,

$$\frac{\partial^2 T}{\partial x^2}$$

GENERAL DIFFERENTIAL EQUATION FOR HEAT CONDUCTION IN CYLINDRICAL COORDINATES

Eq. 9 derived earlier is suitable to analyse heat transfer in regular bodies of rectangular, square or parallelepiped shapes. But, if we have to analyse heat transfer in cylindrical-shaped bodies (which are commonly used in practice), then, working with cylindrical coordinates is more convenient, since in that case, the coordinate axes match with the system boundaries.

Nomenclature for cylindrical coordinate system is shown in Fig. 2.

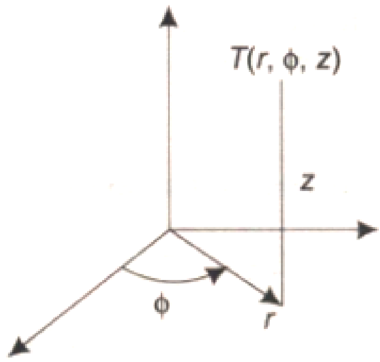


Figure2

Differential equation for heat conduction in cylindrical coordinates may be derived by considering an elemental cylindrical control volume of thickness *dr* and making an energy balance over this control volume, as was done in the case of Cartesian coordinates, or, coordinates transformation can be adopted; for this purpose, transformation equations are,

$$x = r\,cos\emptyset$$
$$y = r\,sin\emptyset$$
$$z = z$$
$$\emptyset = tan^{-1}(y/x)$$

The resulting general differential equation in cylindrical coordinates is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \ldots\ldots\ldots\ldots 10$$

Eq. 10 is the general differential equation in cylindrical coordinates, for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.

For one-dimensional conduction in r direction only, we get from Eq. 10,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \ldots\ldots\ldots\ldots 11$$

And for steady state we get:

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} + \frac{q_g}{k} = 0 \ldots\ldots\ldots .12$$

GENERAL DIFFERENTIAL EQUATION FOR HEAT CONDUCTION IN SPHERICAL COORDINATES

To analyse heat transfer in spherical systems, working with spherical coordinates is more convenient, since the coordinate axes match with system boundaries. Nomenclature for the spherical coordinates is shown in Fig. 3.

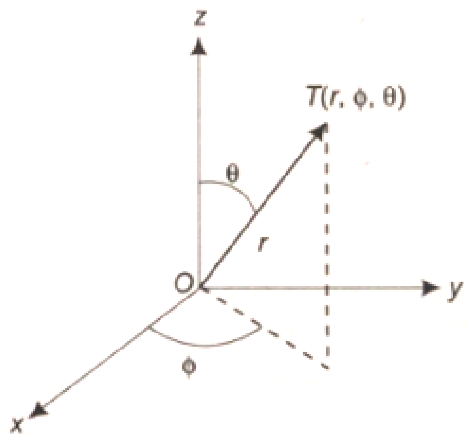


Figure (3)

Differential equation for heat conduction in spherical coordinates may be derived by considering an elemental spherical control volume and making an energy balance over this control volume, as was done in the case of Cartesian and cylindrical coordinates, or, coordinate transformation can be adopted using the following transformation equations,

$$x = r\, sin\theta\, cos\emptyset$$
$$y = r\, sin\theta\, sin\emptyset$$
$$z = r\, cos\theta$$

The resulting general differential equation in spherical coordinates is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\,\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 sin\theta}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \ldots\ldots..13$$

Eq. 13 is the general differential equation in spherical coordinates, for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.

For one-dimensional conduction in r direction only, we get from Eq. 13,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \ldots\ldots\ldots\ldots 14$$

Eq. 14 represents one-dimensional, time dependent conduction in r direction only, with constant k and uniform internal heat generation, in spherical coordinates.

And, for steady state, one-dimensional heat conduction in r direction only, with constant k and uniform heat generation Eq. 14 reduces to,

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = 0$$

ONE-DIMENSIONAL, STEADY STATE HEAT CONDUCTION, WITH NO INTERNAL HEAT GENERATION

a) Plane slab

The governing equation for a plane slab with One-dimensional, steady state heat conduction, with no internal heat generation is:

$$\frac{\partial^2 T}{\partial x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

Integrating Eq.1 once:

$$\frac{\partial T}{\partial x} = C_1$$

Integrating again:

$$T(x) = C_1 x + C_2 \ldots\ldots\ldots\ldots\ldots .2$$

Equation 2 is the general solution for the temperature distribution. Values of the two integration constants C_1 and C_2 are obtained from the two boundary conditions namely

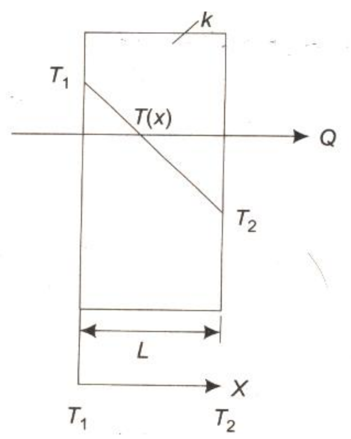


Figure (1)

B.C.(i): $T = T_1 \quad at \quad x = 0$

B.C.(ii): $T = T_2 \quad at \quad x = L$

From B.C.(i) and Eq.2

$$T(0) = T_1 = C_2$$

From B.C.(ii) and Eq.2

$$T(L) = T_2 = C_1 L + C_2$$

$$T_2 = C_1 L + T_1$$

So:

$$C_1 = (T_2 - T_1)/L$$

Sub. values of C_1 and C_2 1n Eq.2 we get,

$$T(x) = \frac{(T_2 - T_1)}{L} x + T_1 \dots\dots\dots\dots\dots\dots 3$$

Eq. 3 can be written in non dimensional form:

$$\frac{T(x) - T_1}{(T_2 - T_1)} = \frac{x}{L} \dots\dots\dots\dots\dots\dots..4$$

To find the heat flux, apply Fourier law,

$$q = -k \frac{dT}{dx}$$

$$\frac{dT}{dx} = C_1 = (T_2 - T_1)/L$$

So:

$$q = -k \frac{(T_2 - T_1)}{L} = k \frac{(T_1 - T_2)}{L} \quad W/m^2 \dots\dots\dots\dots..5$$

The heat flow rate is

$$Q = qA = -kA \frac{(T_2 - T_1)}{L} \qquad W \dots\dots\dots\dots..6$$

b) Cylindrical systems

The governing equation for the Cylindrical systems, One-dimensional, steady state heat conduction, with no internal heat generation is:

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} = 0$$

Multiplying by r, we get.

$$\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2} = 0$$

i.e.

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

Integrating,

$$r\frac{\partial T}{\partial r} = C_1$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

Integrating again,

$$T(r) = C_1 \ \ln(r) + C_2 \dots\dots\dots.a$$

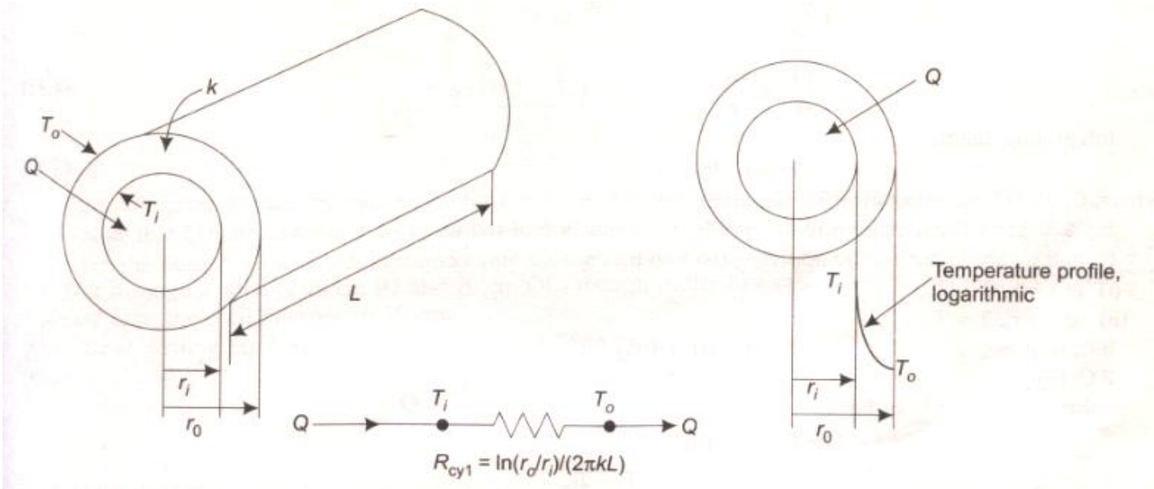


Figure (1b)

The equation gives the temperature distribution as a function of radius.

Where C_1 and C_2 are constants of integration, and they are found out by applying the two B.C:

1- $at \quad r = r_i \quad T = T_i$

2- $at \quad r = r_o \quad T = T_0$

B.C. (1) gives, $T(i) = C_1 \ln(r_i) + C_2 \ldots\ldots b$

B.C. (2) gives, $T(o) = C_1 \ln(r_o) + C_2 \ldots\ldots c$

Subtracting Eq. c from Eq. b:

$$T_i - T_o = C_1 \ln(r_i/r_o)$$

$$C_1 = \frac{T_i - T_o}{\ln(r_i/r_o)} = \frac{T_o - T_i}{\ln(r_o/r_i)}$$

$$C_2 = T_i - \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r_i)$$

Substituting C1 and C2 in equation a, we get

$$T(r) = \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r) + T_i - \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r_i)$$

$$T(r) = T_i + \frac{T_o - T_i}{\ln(r_o/r_i)} \ln\left(\frac{r}{r_i}\right)$$

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

To find the heat transfer rate:

$$\left[Q = -kA_r \frac{dT}{dr}\right]_{r=r_i} = -k2\pi r_i L \frac{C_1}{r_i}$$

$$Q = -k2\pi r_i L \frac{T_o - T_i}{r_i \ln\left(\frac{r_o}{r_i}\right)}$$

$$Q = k2\pi L \frac{T_i - T_o}{\ln\left(\frac{r_o}{r_i}\right)} \ldots \ldots . . d$$

c) Spherical Systems

Spherical system is one of the most commonly used geometries in industry. It finds its applications as storage tanks, reactors, etc. in petrochemical, refineries and cryogenic industries. Sphere has minimum surface area for a given volume and material requirement to manufacture a sphere is minimum compared to other geometries.

The general differential equation in spherical coordinates, for steady state, one-dimensional heat conduction in r direction only, with constant k, no heat generation and uniform heat generation is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = 0$$

The non-dimensional form for the temperature distribution equation is:

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\frac{1}{r} - \frac{1}{r_i}}{\frac{1}{r_o} - \frac{1}{r_i}} = \frac{r_o}{r} * \left(\frac{r - r_i}{r_o - r_i}\right)$$

The heat transfer rate equation is:

$$Q = \frac{\Delta T}{R_{sph}} = \frac{T_i - T_o}{\frac{r_o - r_i}{4\pi k r_o r_i}}$$

Heat flow through composite slabs

Heat transfer through a composite slab, consisting of 2 or 3 layers of materials of different thermal conductivities. This is a very common application, e.g. in the case of insulation of furnace walls, insulation of walls of buildings, refrigerators, cold storage plants, hot water tanks, etc.

While solving heat transfer problems in composite slabs under steady state conditions, it is convenient to use the thermal resistance concept.

Consider a composite slab consisting of three layers 1,2 and 3 as shown in Fig.2. Let the thicknesses of the three layers be L_1, L_2 and L_3, respectively; also, the respective thermal conductivities are k_1, k_2 and k_3.

Fluid at a temperature T_a flows on the surface with a convective heat transfer coefficient of h_a and, a fluid at a temperature of T_b flows with a convective heat transfer coefficient of h_b, as shown. Let T_a be higher than T_b, so that steady state heat transfer rate Q is from left to right as indicated in the Fig. 2.

Assumptions:

1. Steady state, one-dimensional heat conduction.
2. No internal heat generation.
3. Constant thermal conductivities k_1, k_2 and k_3.
4. There is perfect thermal contact between layers, i.e. there is no temperature drop at the interface and the temperature profile is continuous.

Since it is a case of steady state conduction with no internal heat generation, it is clear from the First law that heat flow rate Q, through each layer is the same. Referring to Fig.2, it may be seen that heat flows from the fluid at temperature T_a to the left surface of slab 1 by convection, then by conduction through slabs 1, 2 and 3, and then, by convection from the right surface of slab 3 to the fluid at temperature T_b.
Let the area of the slab normal to the heat flow direction be A(m^2). Now, considering each case by turn.

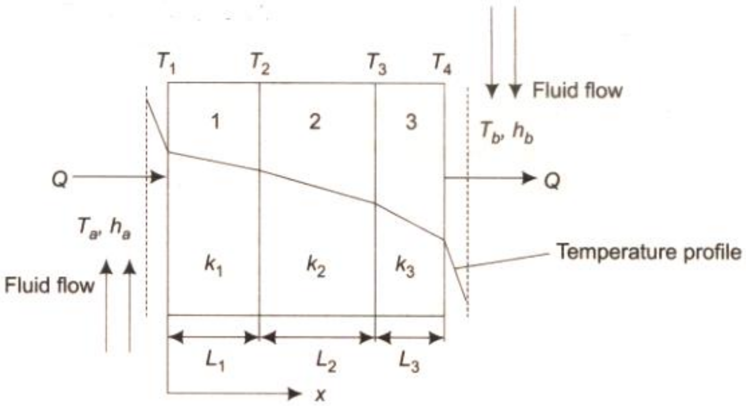


Figure (2)

Convection on the left surface of slab1

$$Q = h_a A(T_a - T_1)$$

So

$$T_a - T_1 = \frac{Q}{h_a A} \dots\dots\dots\dots a$$

Conduction through slab1

$$Q = \frac{k_1 A(T_1 - T_2)}{L_1}$$

So

$$(T_1 - T_2) = \frac{Q L_1}{k_1 A} \dots\dots\dots\dots\dots b$$

Conduction through slab2

$$Q = \frac{k_2 A(T_2 - T_3)}{L_2}$$

So

$$(T_2 - T_3) = \frac{Q L_2}{k_2 A} \dots\dots\dots\dots\dots c$$

Conduction through slab3

$$Q = \frac{k_3 A(T_3 - T_4)}{L_3}$$

So

$$(T_3 - T_4) = \frac{Q L_3}{k_3 A} \dots\dots\dots\dots\dots d$$

Convection on the left surface of slab3

$$Q = h_b A(T_4 - T_b)$$

So

$$T_4 - T_b = \frac{Q}{h_b A} \ldots\ldots\ldots\ldots e$$

Adding Eq. a, b, c, d and e, we get:

$$T_a - T_b = Q\left[\frac{1}{h_a A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_b A}\right] \ldots\ldots\ldots\ldots f$$

i.e.

$$T_a - T_b = Q[R_a + R_1 + R_2 + R_3 + R_b] \ldots\ldots\ldots\ldots g$$

where,

R_a = convective resistance at left surface of slab 1,
R_1 = conductive resistance of slab 1,
R_2 = conductive resistance of slab 2,
R_3 = conductive resistance of slab 3, and
R_b = convective resistance at right surface of slab 3.

So, we write Eq. g as:

$$Q = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b}\right] \ldots\ldots\ldots\ldots 7$$

Now, observe the analogy with Ohm's law. Refer to the Fig.3 for the equivalent thermal circuit. It is clear that (T_a - T_b) is the total temperature potential, Q is the heat current flowing and the total resistance is the sum of the individual five resistances which are in series.

For thermal resistances in series, we have,

$$R_{tot} = \sum R \ldots\ldots\ldots\ldots\ldots 8$$

For thermal resistances in parallel

Thermal resistances may be arranged in parallel too, as shown in Fig.3.

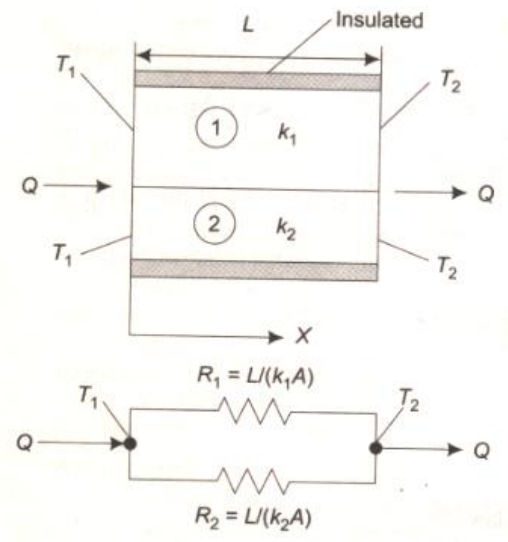


Figure (3)

From the analogy with the electrical circuit, when the resistances are in parallel, the total resistance is given by:

$$\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{\frac{L}{k_1 A}} + \frac{1}{\frac{L}{k_2 A}}$$

$$R_{tot} = \frac{R_1 R_2}{R_1 + R_2} \ldots\ldots\ldots\ldots 9$$

For thermal resistances in series and parallel: general case of thermal resistances arranged in series and parallel is shown in fig. 4.

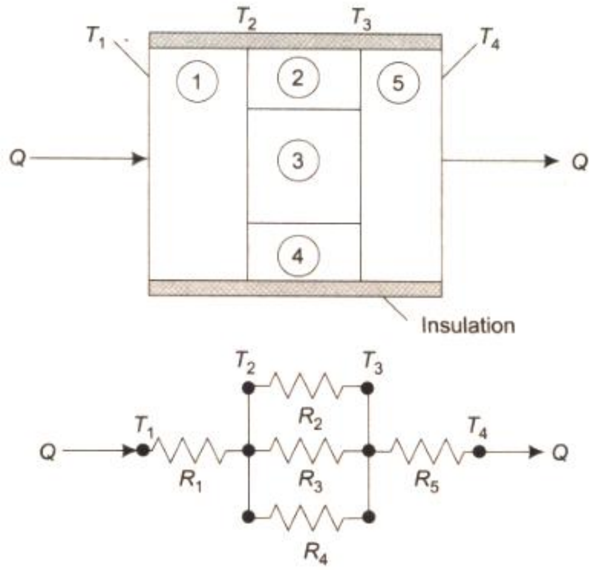


Figure (4)

Applying the rules of electrical circuit for series and parallel, we have,

$$Q = \frac{\Delta T}{R_1 + R_{eff} + R_2} = \frac{T_1 - T_4}{R_1 + R_{eff} + R_2} \ldots\ldots\ldots .10$$

Where R_{eff} is the effective resistance of the three resistances R2, R3 and R4 in parallel as shown in fig. 4.

$$\frac{1}{R_{eff}} = \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \ldots\ldots\ldots\ldots\ldots ..11$$

Thermal resistance for conduction for cylindrical shell

From Eq. d

$$Q = k2\pi L \frac{T_i - T_o}{\ln\left(\frac{r_o}{r_i}\right)} \ldots\ldots .. d$$

Writing Eq. d in a form analogous to ohm's law:

$$Q = \frac{\Delta T}{R_{cyl}} = \frac{T_i - T_o}{\frac{\ln\left(\frac{r_o}{r_i}\right)}{k2\pi L}}$$

we observe that thermal resistance for conduction for cylindrical shell is given by,

$$R_{cyl} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL} \ldots\ldots\ldots\ldots\ldots 12$$

Thermal resistance for spherical system

$$Q = \frac{\Delta T}{R_{sph}} = \frac{T_i - T_o}{\frac{r_o - r_i}{4\pi k r_o r_i}}$$

$$R_{sph} = \frac{r_o - r_i}{4\pi k r_o r_i} = \frac{1}{4\pi k}\left[\frac{1}{r_i} - \frac{1}{r_o}\right]$$

OVERALL HEAT TRANSFER COEFFICIENT, U (W/(M^{2}C))

Consider the case of a furnace where heat is transferred by the hot gases to the inside surface by convection, then by conduction through one, two or three layers of brick and insulation, and finally to ambient air by convection at the outermost surface. This situation is represented in Fig. 2.

Now, in most of the practical cases, temperature of the hot gases (T_a) and that of the ambient (T_b) are known; intermediate temperatures are not known. *We would like to have the heat transfer given by a simple relation of the form:*

$$Q = UA(T_a - T_b) = UA\Delta T \ldots\ldots\ldots 13$$

where, Q is the heat transfer rate (W), A is the area of heat transfer perpendicular to the direction of heat transfer, and $(T_a - T_b) = \Delta T$ is the overall temperature difference.

Our problem is to derive a relation for U.

Now, we have from Eq. 7,

$$Q = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b}\right] \ldots\ldots\ldots\ldots..7$$

Comparing Eq. 7 and Eq. 13;

$$Q = UA(T_a - T_b) = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b}\right] = \frac{T_a - T_b}{\sum R_{th}}$$

So

$$UA = \frac{1}{\sum R_{th}} \ldots\ldots\ldots 14$$

$$U = \frac{1}{A\sum R_{th}} \qquad , \frac{W}{m^2 C} \ldots\ldots\ldots 15$$

$$U = \frac{1}{\frac{1}{h_a} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} + \frac{1}{h_b}} \ldots\ldots..16$$

Remember the expression for U as given by Eq. 15; it is easier and is applicable when we deal with other geometries, too.

Concept of overall heat transfer coefficient is particularly useful in heat exchanger designs. Consider a heat exchanger where a hot fluid flows on one side of a heat exchanger wall and a cold fluid flows on the other side. Then, heat transfer is by convection on the hot side, by conduction across the separating wall and again by convection on the cold side, overall heat transfer coefficient is obtained by applying Eq. 15.

Values of overall heat transfer coefficient for many practical cases are tabulated in handbooks.

Overall Heat Transfer Coefficient for the Cylindrical System

Referring to Fig. 5, it is clear that heat transfer occurs from hot fluid at T_a to the inner cylinder by convection, then through the inner and outer cylindrical shells by conduction and then to the outer cold fluid at T_b by convection. We would rather like to write the heat transfer rate in terms of the known overall temperature difference, as follows,

$$Q = UA(T_a - T_b) = UA\Delta T$$

where U is an overall heat transfer coefficient and A is the area normal to the direction of heat flow. In the case of a plane slab, A was a constant with x; however, in the case of a cylindrical system, area normal to the direction of heat flow is $2\pi rL$, and clearly, this varies with r. Therefore, while dealing with cylindrical systems, we have to specify as to which area U is based on, i.e. whether it is based on inside area or outside area. (Generally, U is based on outside area since pipes are specified on outside diameters.) We write.

$$Q = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

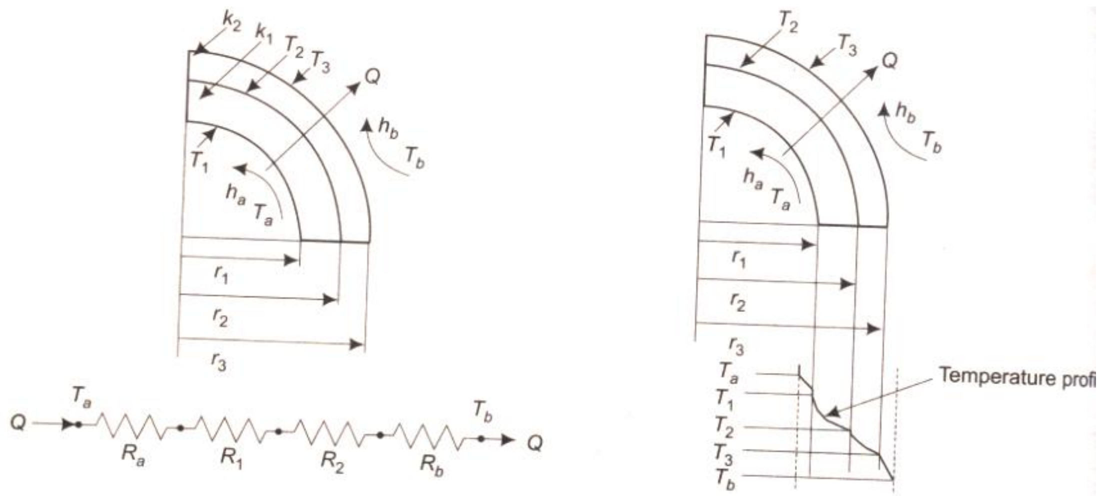


Figure (5)

where,

U_i = overall heat transfer coefficient based on inside area

U_o = overall heat transfer coefficient based on outside area

A_i = heat transfer area on inside

A_o = heat transfer area on outside

Comparing with Eq. 7, we get,

$$Q = \frac{T_a - T_b}{\sum R_{th}} = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

So

$$U_i A_i = U_o A_o = \frac{1}{\sum R_{th}} \dots \dots \dots .17$$

Therefore

$$U_i = \frac{1}{A_i \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}}$$

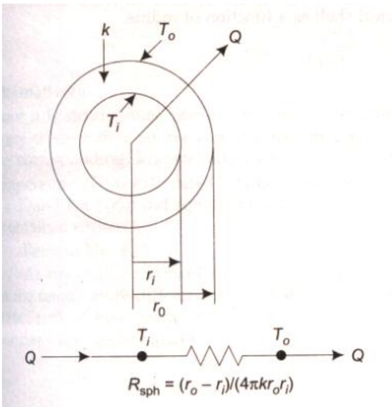
$$U_o = \frac{1}{A_o \sum R_{th}} = \frac{1}{2\pi r_3 L * \left[\frac{1}{2\pi r_1 L h_a} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_1 L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2 L} + \frac{1}{2\pi r_3 L h_b}\right]}$$

$$U_o = \frac{1}{\left[\left(\frac{r_3}{r_1}\right)\frac{1}{h_a} + \left(\frac{r_3}{k_1}\right)\ln\left(\frac{r_2}{r_1}\right) + \left(\frac{r_3}{k_2}\right)\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_b}\right]}$$

Once the total thermal resistance $\sum R$ is calculated, U_i, or U_o is easily found out from Eq.17. The concept of overall heat transfer coefficient in cylindrical systems is often useful in heat exchanger designs, since cylindrical geometry is a popular choice in heat exchangers.

Overall Heat Transfer Coefficient for the spherical System

$$Q = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$



where,
U_i = overall heat transfer coefficient based on inside area
U_o = overall heat transfer coefficient based on outside area
A_i = heat transfer area on inside
A_o = heat transfer area on outside

$$Q = \frac{T_a - T_b}{\sum R_{th}} = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

So

$$U_i A_i = U_o A_o = \frac{1}{\sum R_{th}}$$

Therefore

$$U_i = \frac{1}{A_i \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}} = \frac{1}{4\pi r_1^2 * \left[\frac{1}{4\pi h_a r_1^2} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{4\pi h_b r_3^2} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2 L} + \frac{1}{2\pi r_3 L h_b}\right]}$$

$$U_o = \frac{1}{\left[\frac{1}{h_a}\left(\frac{r_3}{r_1}\right)^2 + \frac{1}{h_b} + \frac{r_3^2 (r_2 - r_1)}{k_1 r_1 r_2} + \frac{r_3 (r_3 - r_2)}{k_2 r_2}\right]}$$

Note: the above calculations give U_i and U_o in term of inside and outside radii. You need not to memorize them. To calculate U_i or Uo while solving numerical problem just remember Eq. 17

CRITICAL THICKNESS OF INSULATION

Let us consider a layer of insulation which might be installed around a circular pipe, as shown in Figure 7. The inner temperature of the insulation is fixed at *Ti*, and the outer surface is exposed to a convection environment at *T*∞. From the thermal network the heat transfer is

$$q = \frac{2\pi L(T_i - T_\infty)}{\frac{\ln(r_o/r_i)}{k} + \frac{1}{r_o h}}$$

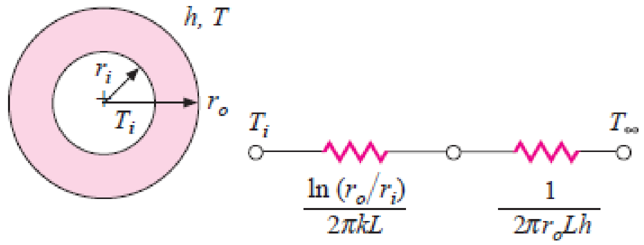


Figure (7)

Now let us manipulate this expression to determine the outer radius of insulation r_o, which will maximize the heat transfer. The maximization condition is

$$\frac{dq}{dr^o} = 0 = \frac{-2\pi L(T_i - T_\infty)\left(\frac{1}{r_o k} - \frac{1}{r_o^2\, h}\right)}{\left[\frac{\ln(r_o/r_i)}{k} + \frac{1}{r_o h}\right]^2}$$

which gives the result

$$r_o = \frac{k}{h}$$

The last equation expresses the critical-radius-of-insulation concept. If the outer radius is less than the value given by this equation, then the heat transfer will be **increased** by adding more insulation. For outer radii greater than the critical value an increase in insulation thickness will cause a **decrease** in heat transfer. The central concept is that for sufficiently small values of *h* the convection heat loss may actually increase with the addition of insulation because of increased surface area.

ONE-DIMENSIONAL, STEADY STATE HEAT CONDUCTION, WITH INTERNAL HEAT GENERATION

A number of interesting applications of the principles of heat transfer are concerned with systems in which heat may be generated internally. Nuclear reactors are one example; electrical conductors and chemically reacting systems are others.

Plane slab with Heat Sources

Consider a plane slab of thickness 2L as shown in Fig. 8. Other dimensions of the slab are comparatively large, so that heat transfer may be considered as one-dimensional in the x-direction, as shown.

The slab has a constant thermal conductivity k, and a uniform internal heat generation rate of q_g (W/m^3). Both the sides of the slab are maintained at the same, uniform temperature of T_w. Then, it is clear that maximum temperature will occur at the centre line, since the heat has to flow from the centre outwards. Therefore it is advantageous to select the origin of the rectangular coordinate system on the centre line, as shown.

Let us analyse this case for temperature distribution within the slab and the heat transfer to the sides.

Assumptions:

1. One-dimensional conduction i.e. thickness L is small compared to the dimensions in the y and z directions.
2. Steady state conduction, i.e. temperature at any point within the slab does not change with time: of course, temperatures at different points within the slab will be different.
3. Uniform internal heat generation rate, q_g (W/m^3)

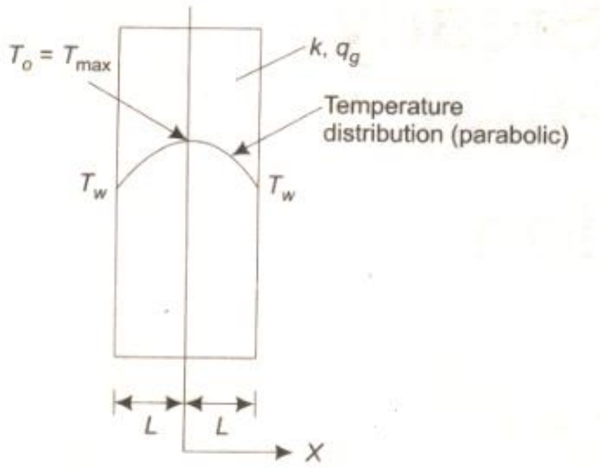


Figure (8)

4. Material of the slab is homogeneous (i.e. constant density) and isotropic (i.e. value of k is same in all directions).

We wish to find out the temperature field within the slab and then the heat flux at any point

We start with the general differential equation in Cartesian coordinates, for the above-mentioned assumptions.

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_g}{k} = 0 \ldots\ldots .1.1$$

B.C.:

1) $at\ x = 0, dT/dx = 0$since the temperature is maximum at the center line.
2) $at\ x = \mp L,\ T = T_w$

Integrating Eq. 1.1

$$\frac{dT}{dx} = \frac{-q_g x}{k} + C_1 \ldots\ldots\ldots\ldots .a$$

Integrating again,

$$T(x) = \frac{-q_g x^2}{2k} + C_1 x + C_2 \ldots\ldots\ldots .1.2$$

Applying B.C. (1) to Eq. a:

$$C_1 = 0$$

Applying B.C. (2) to Eq. 1.2:

$$T_w = \frac{-q_g L^2}{2k} + C_2$$

$$C_2 = T_w + \frac{q_g L^2}{2k}$$

Sub. C_1 and C_2in Eq. 1.2:

$$T(x) = \frac{-q_g x^2}{2k} + T_w + \frac{q_g L^2}{2k}$$

$$T(x) = T_w + \frac{q_g}{2k}(L^2 - x^2) \ldots\ldots\ldots .1.3$$

Where, **L is half thickness of the slab** (Remember this)

Also by observation, $T = T_{max} \quad at\ \ x = 0$.

Then putting $x = 0$ in Eq. 1.3:

$$T_{max} = T_w + \frac{q_g L^2}{2k} \ldots\ldots\ldots\ldots .1.4$$

Then from Eq. 1.3 and Eq. 1.4, we get:

$$\frac{T - T_w}{T_{max} - T_w} = \frac{L^2 - x^2}{L^2} = 1 - \left(\frac{x}{L}\right)^2$$

Heat transfer

In the case of a slab with no internal heat generation, heat flux was the same at every point within the slab, since dT/dx was a constant and independent of x. However, when there is heat generation, dT/dx is not independent of x (see Eq. a), and obviously, heat flux , $q\ (= -kAdT/dx)$ varies from point to point along x within the slab, for the heat transfer at both the surfaces.

$$Q = -kA(dT/dx) \quad at\ x = L$$

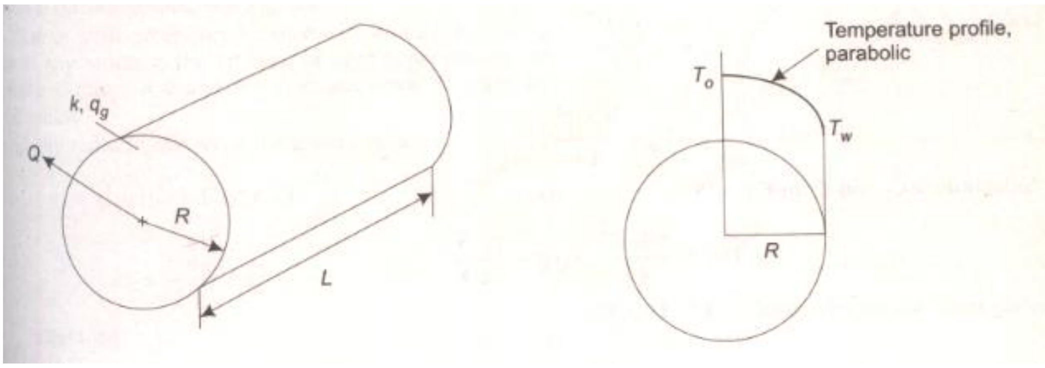
By substituting Eq. a, we get:

$$Q = -kA\left[\frac{-q_g x}{k}\right] \quad at\ x = L$$

$$Q = +q_g AL$$

CYLINDER WITH HEAT SOURCES

consider a solid cylinder of radius, R and length, L. There is uniform heat generation within its volume at a rate of q_g (W/m^3). Let the thermal conductivity, k be constant. See the Figure.



We would like to analyse this system for temperature distribution and maximum temperature attained.

Assumption:

1. Steady state conduction
2. one-dimension conduction, in the r direction only
3. Homogeneous, isotropic material with constant k
4. Uniform internal heat generation rate. q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates reduces to:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{q_g}{k}=0 \ldots\ldots..b$$

Multiplying by r.

$$r\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{\partial r}=\frac{-q_g r}{k}$$

Integrating:

$$r\frac{\partial T}{\partial r}=\frac{-q_g r^2}{2k}+C_1$$

$$\frac{\partial T}{\partial r}=\frac{-q_g r}{2k}+\frac{C_1}{r}\ldots\ldots\ldots..c$$

Integrating again:

$$T(r)=\frac{-q_g r^{\,2}}{4k}+C_1\ln(r)+C_2 \ldots\ldots\ldots\ldots\ldots..1.5$$

C1 and C2, the constants of integration are obtained by applying the boundary conditions B.C. are:

1) $at\ r=0, dT/dr=0$ i.e. at the temperature at the center of the cylinder, temperature is finite and maximum (T_o = T_{max}) because heat flow from inside to outside.
2) $at\ r=R\ i.e. at\ the\ surfuce, T=T_w$

From B.C.1 and Eq. c, we get: $C_1=0$

From B.C.2 and Eq. 1.5, we get:

$$T_w=\frac{-q_g R^{\,2}}{4k}+C_2$$

$$C_2=T_w+\frac{q_g R^{\,2}}{4k}$$

Sub. $C_1\ and\ C_2\ in\ Eq.$ 1.5:

$$T(r)=\frac{-q_g r^{\,2}}{4k}+T_w+\frac{q_g R^{\,2}}{4k}$$

$$T(r)=T_w+\frac{q_g}{4k}\left(R^{\,2}-r^{\,2}\right)\ldots\ldots.1.6$$

Maximum temperature occurs at the centre, because of symmetry considerations (i.e. heat flows from the center radially outward in all directions; therefore, temperature at the centre must be a maximum.)

Therefore, putting r = 0 in Eq. 1.6:

$$T_{max} = T_w + \frac{q_g R^{\,2}}{4k} \dots \dots \dots 1.7$$

From Eq. 1.6 and 1.7,

$$\frac{T - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2$$

HEAT TRANSFER FROM EXTENDED SURFACES (FINS)

Introduction

Fins are generally used to enhance the heat transfer from a given surface.

Consider a surface losing heat to the surroundings by convection. Then, the heat transfer rate Q, is given by Newton's law of cooling:

$$Q = h\,A\,(T_o - T_a)$$

Where,

h = heat transfer coefficient between the surface and the ambient

A = exposed area of the surface

T_o = temperature of the surface, and

T_a = temperature of the surroundings.

So if we need to increase the heat transfer rate from the surface, we can:

1. increase the temperature potential, (To - Ta); but, this may not be possible always since both these temperature may not be in our control.
2. increase the heat transfer coefficient h; this also may not be always possible or it may need installing an external fan or pump to increase the fluid velocity and this may involve cost consideration, or
3. increase the surface area A; in fact, this is the solution generally adopted. Surface area is increased by adding an 'extended surface' (or, fin) to the 'base surface' by extruding, welding or by simply fixing it mechanically.

Adding of fins can increase the heat transfer from the surface by several folds, e.g. an automobile radiator has thin sheets fixed over the tubes to increase the area several folds and thus increase the rate of heat transfer

Generally, fins are fixed on that side of the surface when the heat transfer coefficient is low. Heat transfer coefficient are lower for gases as compared to liquids. Therefore, one can observe that fins are fixed on the outside of the tubes in a car radiator, where cooling liquid flows inside the tubes and air flows on the outside across fins.

Likewise, in the condenser of a household refrigerator, freon flows inside the tubes and the fins are fixed on the outside of these tubes to enhance the heat transfer rate.

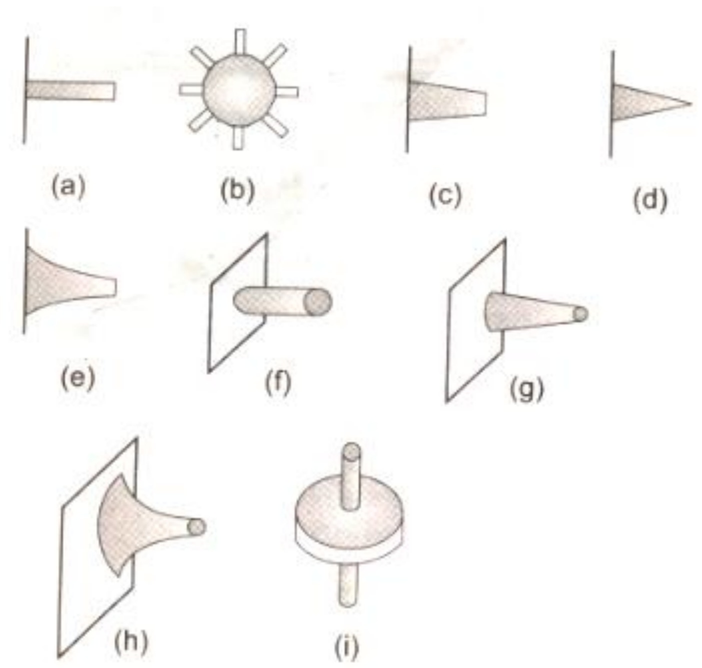
Application areas of fins are:

1. Radiators for automobiles
2. Heat exchangers of a wide variety, used in different industries
3. Cooling of electric motors, transformers, etc.
4. Cooling of electronic equipments, chips, I.C. boards etc.

Types of fins:

There are innumerable types of fins used in practice some of the more common types are shown in Fig.

Fins of rectangular(a), circular, triangular(d), trapezoidal(c) and conical(g) sections are some of the types commonly used.



Determination of heat transfer in fins requires information about the temperature profile in the fin. We get the differential equation describing the temperature distribution in the fin by the usual procedure of writing an energy balance for a differential volume of the fin.

Fins of Uniform Cross Section (Rectangular or Circular)-Governing Differential Equation

Let us analyst' heat transfer in a fin of rectangular cross section. Same analysis will be valid for a fin of circular cross section also.

Consider a fin of rectangular cross section attached to the base surface, as shown in Fig.

Let L be the length of fin,

w, its width and

t its thickness.

Let P be the perimeter = 2 (w + t).

Let A_c be the area of cross section and

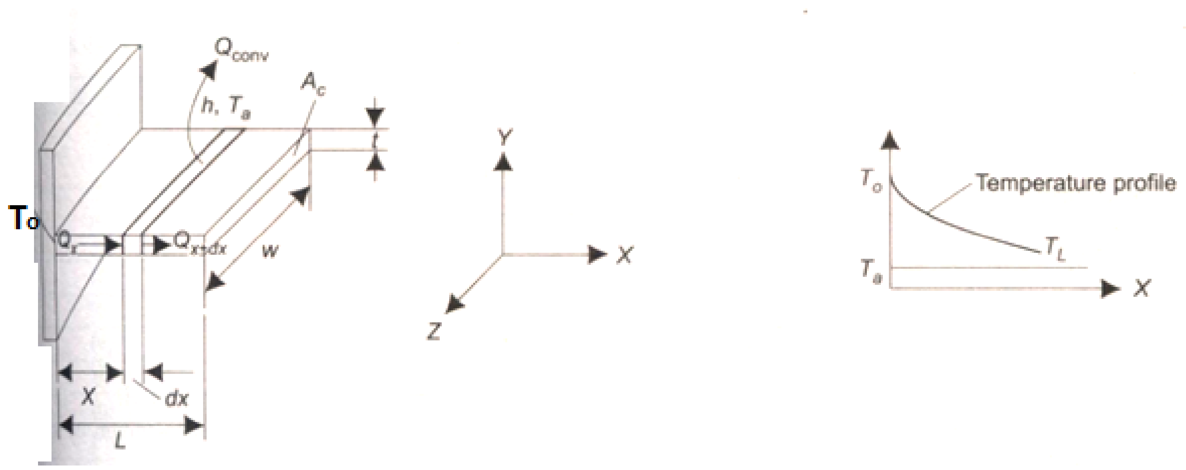
T_o the temperature at the base, as shown.

Assumptions:

1. Steady state conduction, with no heat generation in the fin
2. Thickness t is small compared to length L and width w, i.e. one-dimensional conduction in the X-direction only.
3. Thermal conductivity, k of the fin material is constant.
4. Uniform heat transfer coefficient h, over the entire length of fin.

5. No bond resistance in the joint between the fin and the base wall, and
6. Negligible radiation effect.

Base temperature, T_o is higher than the ambient temperature, T_a Temperature will drop along the fin from the base to the tip of the fin, as shown in Fig. Heat transfer will occur by conduction along the length of the fin and by convection, with a heat transfer coefficient h, from the surface of the fin to the ambient.
Our aim is to derive a differential equation governing the temperature distribution in the fin. Once we get the temperature field, heat flux at any point can easily be obtained by applying Fourier's law.



consider an elemental section of thickness dx at a distance x from the base as shown. Let us write an energy balance for this element:
Energy going into the element by conduction = (Energy leaving the element by conduction) + Energy leaving the surface of the element by convection)

$$Q_x = Q_{x+dx} + Q_{conv} \;\dots\dots\dots a$$

Q_x = heat conducted into the element at x
Q_{x+dx} = heat conducted out of the element at x + dx, and
Q_{conv} = heat convected from the surface of the element to ambient

We have:

$$Q_x = -kA_c \frac{\partial T}{\partial x}$$

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} dx$$

$$where: \qquad Q_x = -kA_c \frac{\partial T}{\partial x} \qquad Fourier's\ law\ sub.in\ last\ equation$$

$$Q_{x+dx} = -kA_c \frac{\partial T}{\partial x} - kA_c \frac{\partial^2 T}{\partial x^2} dx$$

$$Q_{conv} = h\, A_s (T - T_a)$$

$$Q_{conv} = h\,(P\,dx)(T - T_a)$$

Where A_s is the surface area of the element, P is the perimeter substituting the terms in Eq. a:

$$-kA_c\frac{\partial T}{\partial x} = \left(-kA_c\frac{\partial T}{\partial x} - kA_c\frac{\partial^2 T}{\partial x^2}dx\right) + h\,(P\,dx)(T - T_a)$$

$$kA_c\frac{\partial^2 T}{\partial x^2}dx - h\,(P\,dx)(T - T_a) = 0$$

$$\frac{\partial^2 T}{\partial x^2} - m^2.(T - T_a) = 0 \ldots\ldots\ldots\ldots b$$

Where

$$m = \sqrt{\frac{h\,.P}{k\,.A_c}}$$

Note that m has units of: (m^{-1}) and is a constant, since for a given operating conditions of a fin, generally h and k are assumed to be constant.

Now, define excess temperature,

$$\theta = (T - T_a)$$

Therefore

$$\frac{\partial \theta}{\partial x} = \frac{\partial T}{\partial x} \quad and, \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}$$

Substituting in Eq. b,

$$\frac{\partial^2 \theta}{\partial x^2} - m^2.\theta = 0 \ldots\ldots\ldots..1$$

Eq.1 is the governing differential equation for the fin of uniform cross section considered.

Eq.1 is a second order, linear, ordinary differential equation. Its general solution is given by calculus theory, in two equivalent forms:

$$\theta(x) = C_1.e^{-mx} + C_2.e^{mx} \ldots\ldots\ldots 2.a$$

where, C_1 and C_2 are constants

and,

$$\theta(x) = A\,.\cosh(mx) + B.\sinh(mx) \ldots\ldots\ldots.2.b$$

where A and B are constants, and cosh and sinh are hyperbolic functions, defined in Table 1.

Eq.2a or .2b describes the temperature distribution in the fin along its length.

To calculate the set of constants C_1 and C_2, or A and B, we need two boundary conditions:

One of the B.C.'s is that the temperature of the fin at its base, i.e. at x = 0, is T_o and this is considered as known.

i.e. B.C. (1): at x = 0, T = T_o

Regarding the second boundary condition, there are several possibilities:

Case (1): Infinitely long fin.

Case (2): Fin insulated at its end (i.e. negligible heat loss from the end of the fin), and

Case (3): Fin losing heat from its end by convection.

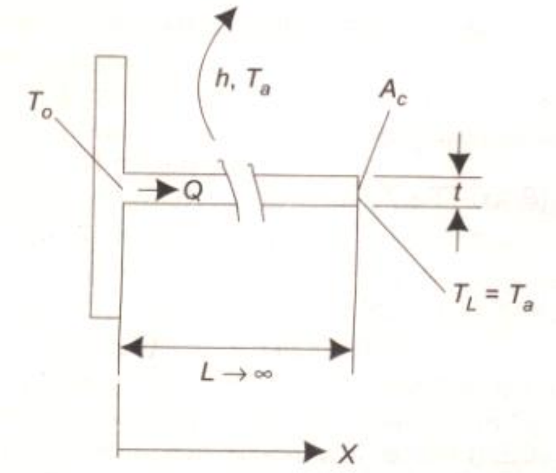
It may be remarked here, that while for case (1), it is convenient to choose the solution in the form given by Eq. 2a and for cases (2) and (3), choosing the solution in the form given by Eq. 2b makes the analysis easy.

TABLE 1 Relations for hyperbolic functions

Sl. No.	Relation
(a)	$\cos h(\beta) = \frac{\exp(\beta) + \exp(-\beta)}{2}$
(b)	$\sin h(\beta) = \frac{\exp(\beta) - \exp(-\beta)}{2}$
(c)	$\exp(\beta) + \exp(-\beta) = 2 \cos h(\beta)$
(d)	$\exp(\beta) - \exp(-\beta) = 2 \sin h(\beta)$
(e)	$\exp(\beta) = \cos h(\beta) + \sin h(\beta)$
(f)	$\exp(-\beta) = \cos h(\beta) - \sin h(\beta)$
(g)	$\sin h(0) = 0$
(h)	$\cos h(0) = 1$
(i)	$\frac{d}{dx}(\sin h(m \cdot x) = m \cdot \cos h(m \cdot x)$
(j)	$\frac{d}{dx} \cos h(m \cdot x) = m \cdot \sin h(m \cdot x)$
(k)	$\cos h(-x) = \cos h(x)$
(l)	$\sin h(-x) = -\sin h(x)$
(m)	$\cos h(x + y) = \cos h(x) \cdot \cos h(y) + \sin h(x) \cdot \sin h(y)$
(n)	$\cos h(x - y) = \cos h(x) \cdot \cos h(y) - \sin h(x) \cdot \sin h(y)$
(o)	$\sin h(x + y) = \sin h(x) \cdot \cos h(y) + \cos h(x) \cdot \sin h(y)$
(p)	$\sin h(x - y) = \sin h(x) \cdot \cos h(y) - \cos h(x) \cdot \sin h(y)$

Case (1) Infinitely Long Fin

This simply means that the fin is very long. Consequence of this assumption is that temperature at the tip of the fin approaches that of the surrounding ambient as the fin length approaches infinity. See Fig.



To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2 \theta}{\partial x^2} - m^2 . \theta = 0 \ldots \ldots \ldots . .1$$

And, we shall choose for its solution for temperature distribution Eq. (2a), i.e.

$$\theta(x) = C_1 . e^{-mx} + C_2 . e^{mx} \ \ldots\ldots\ldots 2.a$$

C1 and C2 are obtained from the B.C.s:

B.C. (1): $at\ x = 0, T = T_o$

B.C. (2): $at\ x \rightarrow \infty, T = T_a$,the ambient temperature.

From B.C. (1):

$$at\ x = 0, \qquad \theta(x) = T_o - T_a = \theta_o$$

From B.C. (2):

$$at\ x = \infty, \qquad \theta(x) = T_a - T_a = 0$$

From B.C. (2) and Eq. 2.a: $C_2 = 0$

From B.C. (1) and Eq. 2.a: $C_1 = \theta_o$

Substituting C_1 and C_2 back in Eq. 2.a, we get:

$$\theta(x) = \theta_o . e^{-mx}$$

$$\frac{\theta(x)}{\theta_o} = e^{-mx}$$

$$\frac{T(x) - T_a}{T_o - T_a} = e^{-mx} \ \ldots\ldots\ldots\ldots 3$$

Eq. 3 gives the temperature distribution in an infinitely long fin of uniform cross section, along the length.

To determine the heat transfer rate:

Hat transfer rate from the fin may be determined by either of the two ways:

a) by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c \frac{\partial T(x)}{\partial x} = -kA_c \frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0 \ldots\ldots c$$

b) by integrating the convective heat transfer for the entire surface of the fin, i.e.

$$Q_{fin} = \int_0^L h.P.(T - T_a)dx = \int_0^L h.P.\theta dx \ \ldots\ldots\ldots.d$$

By method (a):

$$Q_{fin} = -kA_c \frac{\partial T(x)}{\partial x} = -kA_c \frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0$$

$$Q_{fin} = -kA_c \frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0$$

$$Q_{fin} = -kA_c \left[\frac{\partial}{\partial x}(\theta_o . e^{-mx})\right], \qquad at\ x = 0$$

$$Q_{fin} = -kA_c(-m)[\theta_o . e^{-mx}], \qquad at\ x = 0$$

$$Q_{fin} = k.A_c.m.\theta_o \ \ldots\ldots\ldots..4$$

Where

$$m = \sqrt{\frac{h\ .P}{k\ .A_c}}$$

$$Q_{fin} = \sqrt{h\ .P.k.A_c}.\theta_o = \sqrt{h\ .P.k.A_c}.(T_o - T_a) \ \ldots..5$$

Eq. 4 and Eq. 5 gives the heat transfer rate through the fin.
Let us verify this result from method (b):
By method (a):

$$Q_{fin} = \int_0^L h.P.(T - T_a)dx = \int_0^L h.P.\theta dx \dots\dots\dots.d$$

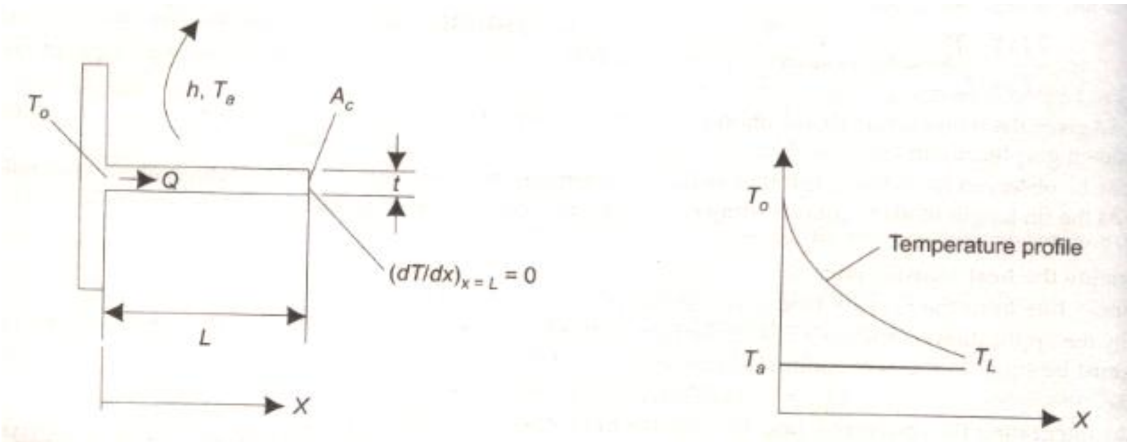
$$Q_{fin} = \int_0^\infty h.P.\theta_o.e^{-mx}\,dx$$

$$Q_{fin} = \frac{1}{m}.h.P.\theta_o$$

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o = \sqrt{h.P.k.A_c}.(T_o - T_a) \quad same\ Eq.5$$

Case (2) Fin of finite length with insulated end

End of a fin is generally not insulated, so here what we mean is that the heat transfer from the end of the fin is negligible as compared to the heat transfer from the surface of the fin. Mostly this is true, since the area of the end of the fin is negligible as compared to the exposed surface area of the fin; in fact this is the most important case. See fig.



To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2\theta}{\partial x^2} - m^2.\theta = 0 \dots\dots\dots..1$$

And, we shall choose for its solution for temperature distribution Eq. (2b), i.e.

$$\theta(x) = A.\cosh(mx) + B.\sinh(mx) \dots\dots\dots.2.b$$

A and B are obtained from the B.C.s:

B.C. (1): $at\ x = 0, T = T_o,\ so\ \theta(x) = T_o - T_a = \theta_o$

B.C. (2): $at\ x = L,\ \frac{dT}{dX} = \frac{d\theta}{dX} = 0$ since the end is insulated.

From B.C. (1) and Eq. 2b:

$$A = \theta_o$$

From B.C. (2) and Eq. 2b:

$$\left(\frac{d\theta}{dX}\right)_{x=L} = 0$$

$$A\,.m.\sinh(mL) \; + B.m.\cosh(mL) \; = 0 \qquad \textit{using relations in table } 1$$

Sub. A

$$\theta_o\,.m.\sinh(mL) \; + B.m.\cosh(mL) \; = 0$$

$$B = -\theta_o \frac{\sinh(mL)}{\cosh(mL)}$$

Substituting for A and B in Eq. 2b

$$\theta(x) = \theta_o\,.\cosh(mx) - \theta_o \frac{\sinh(mL)}{\cosh(mL)}.\sinh(mx)$$

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh(mL)\,.\cosh(mx) - \sinh(mL)\,.\sinh(mx)}{\cosh(mL)}$$

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh(m.(L-x))}{\cosh(mL)} \dots\dots\dots..6, \quad \textit{using relation in table } 1$$

$$\frac{T(x)-T_a}{T_o-T_a} = \frac{\cosh(m.(L-x)))}{\cosh(mL)} \dots\dots\dots..7$$

Eq.6 or 7 gives the temperature distribution in the fin with negligible heat transfer from its end.

Temperature at the end of the fin:

This is easily determined by putting x=L in Eq. 6 or 7

$$\frac{\theta(L)}{\theta_o} = \frac{1}{\cosh(mL)} \dots\dots..6a$$

$$\frac{T_L-T_a}{T_o-T_a} = \frac{1}{\cosh(m.L)} \dots\dots\dots..7a$$

$$T_L = \frac{T_o-T_a}{\cosh(mL)} + T_a \dots\dots 7b$$

Eq. 7b gives the Temperature at the end of this fin (i.e. at x=L)

To determine the heat transfer rate:

Hat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c\left(\frac{dT(x)}{dx}\right)_{x=0} = -kA_c\left(\frac{d\theta(x)}{dx}\right)_{x=0}$$

$$Q_{fin} = -kA_c\theta_o\left(\frac{-m.\sinh(m.(L-x))}{\cosh(m.L)}\right)_{x=0}$$

$$Q_{fin} = k.A_c.m.\theta_o\,.\tanh(m.L) \dots\dots\dots..8$$

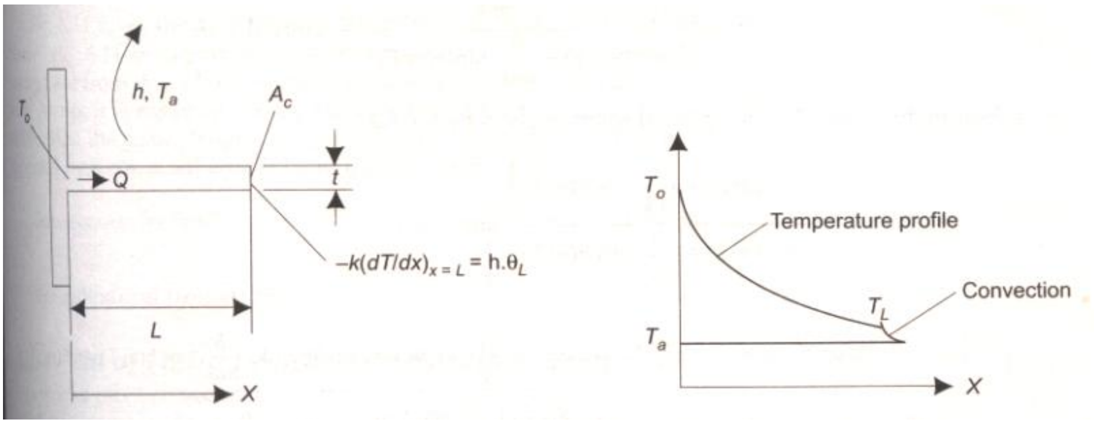
$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o\,.\tanh(m.L) \dots\dots..9$$

Eq.8 or 9 gives the heat transfer rate from the fin, insulated at its end.

Comparing Eq. 8 with that obtained for heat transfer from an infinitely long fin, i.e. Eq. 4, we see that a fin with insulated end becomes equivalent to an infinitely long fin $\tanh(m.L) = 1$.

Case (3) Fin of finite length losing heat from its end by convection

This is a more realistic case, though the relations developed are a little more complicated, see fig.



Here, heat conducted to the tip of the fin must be equal to the heat convected away from the tip to the ambient, i.e.

$$-kA_c\left(\frac{\partial T}{\partial x}\right)_{x=L} = h\,A_c(T_L - T_a)$$

$$-k.\left(\frac{\partial T}{\partial x}\right)_{x=L} = h.\theta_L$$

To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2\theta}{\partial x^2} - m^2.\theta = 0 \dots\dots\dots..1$$

And, we shall choose for its solution for temperature distribution Eq. (2b), i.e.

$$\theta(x) = A\,.\cosh(mx) + B.\sinh(mx) \dots\dots\dots.2.b$$

A and B are obtained from the B.C.s:

B.C. (1): $at\ x\ =\ 0, T\ =\ T_o,\ so\ \theta(x) = T_o - T_a = \theta_o$

Applying B.C. (1) to Eq. 2b:

$$A = \theta_o$$

B.C. (2):

heat conducted to the end = heat convected from the end

$$-kA_c\left(\frac{\partial\theta(x)}{\partial x}\right)_{x=L} = h\,A_c\theta(L), \qquad where\ \theta(L) = T_L - T_a$$

After applying the B.C.s and using Eq. 2b and the relation in table 1, we get:

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh\big(m.(L-x)\big) + \frac{h}{m.k}\sinh(m(L-k))}{\cosh(m.L) + \frac{h}{m.k}\sinh(m.L)} \dots\dots.10$$

Eq. 10 gives the temperature distribution in a fin losing heat by convection at its end.

Note that when $h = 0$, i.e. for negligible heat transfer at the tip of the fin, Eq. 10 reduced to Eq. 6, for a fin with insulated tip.

To determine the heat transfer rate:

Hat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c\left(\frac{dT(x)}{dx}\right)_{x=0} = -kA_c\left(\frac{d\theta(x)}{dx}\right)_{x=0}$$

$$Q_{fin} = kA_c\theta_o m\frac{\tanh(m.L) + \frac{h}{m.k}}{1 + \frac{h}{m.k}.\tanh(m.L)} \dots\dots..11$$

Eq. 11 gives the heat transfer rate from a fin losing heat by convection at its tip.

Note: Eq. 11 is important since it represents the heat transfer rate for a practically important case of a fin loosing heat from its end. However, it is rather complicated to use. So, in practice even when the fin loosing heat from its tip, it is easier to use Eq. 8 and 9 obtained for a fin with insulated tip, but with a correct length, L_c, rather than the actual length, L, to include the effect of convection at the tip. In that case only to evaluate Q, L is replaced by a corrected length L_c, in Eq. 8 and 9, as follows:

For rectangular fins: $L_c = L + \frac{t}{2}$ where t is the thickness of fin

For cylindrical (round) fins: $L_c = L + \frac{r}{2}$ where r is the radius of the cylindrical fin.

Performance of Fins

Recollect that purpose of attaching fins over a surface is to increase the heat transfer rate. How well this purpose is achieved is characterised by two performance parameters:

1) Fin efficiency, η_f, and
2) Fin effectiveness, ε_f.

Fin Efficiency

Fin transfers heat to the surroundings from its surface, by convection. For convection heat transfer, the driving force is the temperature difference between the surface and the surrounding. However, temperature drop along the length of the fin because of the finite thermal conductivity of the fin material; so, heat transfer becomes less effective towards the end of the fin. Obviously, in the ideal case of the entire fin being at the same temperature as that of the base wall, the heat transferred from the fin will be maximum, So fin efficiency is defined as the amount of heat actually transferred by a given fin to the ideal amount of heat that would be transferred if the entire fin were at its base temperature, i.e.

$$\eta_f = \frac{Q_{fin}}{Q_{max}} \dots\dots.12$$

where,

Q_{fin} = actual amount of heat transferred from the fin, and

Q_{max} = maximum (or ideal) amount of heat that would be transferred from the fin, if the entire fin surface were at the temperature of the base,

(a) For an infinitely long fin:

For an infinitely long fin, actual heat transferred is given by Eq. 5:

$$Q_{fin} = \sqrt{h\,.P.k.A_c}.\theta_o = \sqrt{h\,.P.k.A_c}.(T_o - T_a) \; \dots..5$$

To calculate Q_{max}, if the entire fin surface were to be at a temperature of T_o, the convective heat transfer from the surface would be:

$$Q_{max} = h.P.L.(T_o - T_a) \; \dots\dots\dots\dots A$$

where, P is the perimeter of the fin section and (P.L) is the surface area of the fin.

Dividing Eq. 5 by Eq. A:

$$\eta_f = \frac{\sqrt{h\,.P.k.A_c}.(T_o - T_a)}{h.P.L.(T_o - T_a)}$$

$$\eta_f = \frac{1}{\sqrt{\frac{h.P}{k.A_c}}.L}$$

$$\eta_f = \frac{1}{m.L} \; \dots\dots..13$$

(b) For a fin with insulated end:

For the case of a fin with an insulated end, we get actual heat transferred Q_{fin} from Eq. 9:

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o\,.\tanh(m.L) \; \dots\dots..9$$

and, fin efficiency is given by:

$$\eta_f = \frac{\sqrt{h\,.P.k.A_c}.(T_o - T_a)\,.\tanh(m.L)}{h.P.L.(T_o - T_a)}$$

$$\eta_f = \frac{\tanh(m.L)}{\sqrt{\frac{h.P}{k.A_c}}.L}$$

$$\eta_f = \frac{\tanh(m.L)}{m.L} \; \dots\dots 14$$

$$\eta_f = \frac{\tanh(m.L)}{m.L}$$

Note For the more realistic case of a fin losing heat from its end, as stated earlier, to calculate heat transfer, Eq. 9 itself may be used, but, with a corrected length L_c in place of L.

Note: for a thin fin or a very wide fin i.e. w >> t, we can write:

$$m.L=\sqrt{\frac{h.P}{k.A_c}}.L=\sqrt{\frac{h.(2w+2t)}{k.w.t}}.L$$

$$m.L=\sqrt{\frac{2.h.w}{k.w.t}}.L=\sqrt{\frac{2.h}{k.t}}.L$$

Multiplying the numerator and the dominator by $L^{1/2}$ gives:

$$m.L=\sqrt{\frac{2.h}{k.t.L}}.L^{\frac{3}{2}}$$

$$m.L=\sqrt{\frac{2.h}{k.A_m}}.L^{\frac{3}{2}}$$

Where, $A_m = (L.t)$, is the profile area for the rectangular section

Also

$$\frac{P}{A_c}=\frac{(2w+2t)}{wt}=\frac{2w}{wt}=\frac{2}{t}$$

Also

$$m=\sqrt{\frac{h.P}{k.A_c}}=\sqrt{\frac{2.h}{k.t}}$$

Figure 1 | Efficiencies of circumferential fins of rectangular profile, according to Reference 3.

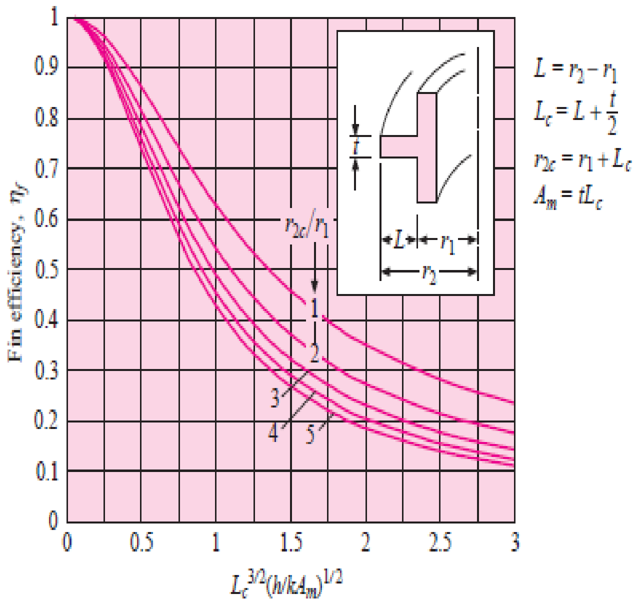
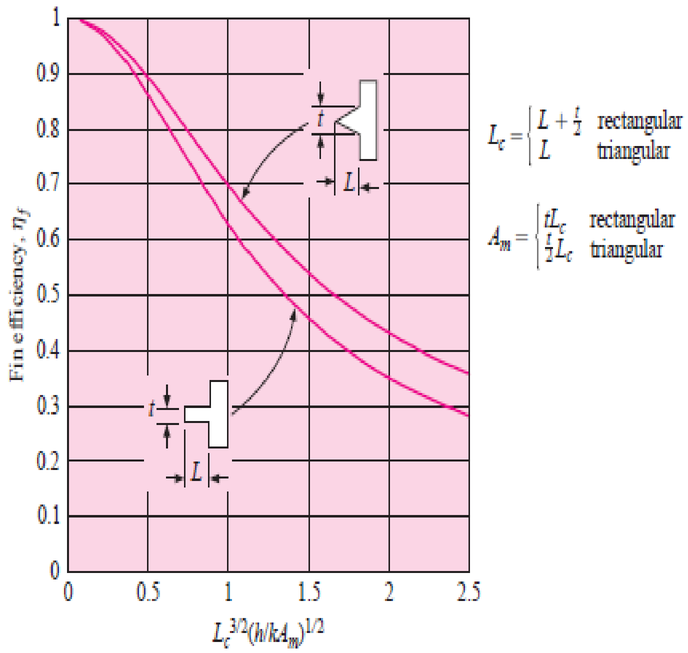


Figure 2 | Efficiencies of straight rectangular and triangular fins.



Fin Effectiveness (ε_f)

Consider a fin of uniform cross-sectional area A_c, fixed to a base surface. Purpose of the fin is to enhance the heat transfer. If the fin were not there, heat would have been transferred from the base area A_c, by convection. By attaching the fin, area for convection increases i.e. convective resistance (1= /(h.A)) decreases; however, it is obvious that a conduction resistance due to the solid fin is now introduced and the total heat transfer would depend upon the net thermal resistance. As we go on increasing the length of fin, convection resistance will go on decreasing, but conduction resistance will go on increasing. This means that attaching a fin may not necessarily result in effectively increasing the heat transfer. Therefore, how effective the fin is in enhancing the heat transfer is characterised by a parameter called fin effectiveness.

Fin effectiveness is defined as the ratio of the heat transfer rate with the fin in place, to the heat transfer that would occur if the fin were not there, from the area of the base surface where the fin was originally fixed.

ε_f = (heat transfer rate with fin)/(heat transfer rate without fin)

$$\varepsilon_f = \frac{Q_{fin}}{h.A_c.(T_o - T_a)} \dots \dots 16$$

Fin effectiveness equal to 1 means that there is no enhancement of heat transfer at all by using the; fin if the fin effectiveness is less than 1, that means that the fin actually reduces the heat transfer by adding additional thermal resistance! Obviously, ε_f should be as large as possible. Use of fins is hardly justified unless fin effectiveness is greater than about 2, i.e. $\varepsilon_f > 2$.

To get an insight into the physical implications of fin effectiveness, let us consider an infinitely long fin

Then, we have:

$$\varepsilon_f = \frac{\sqrt{h\,.P.k.A_c}.(T_o - T_a)}{h.A_c.(T_o - T_a)}$$

$$\varepsilon_f = \sqrt{\frac{k.P}{h.A_c}} \dots \dots \dots 17$$

Eq. 17 is an important equation. Following significant conclusions may be derived from this equation:

1. Thermal conductivity, k should be as high as possible; that is why we see that generally, fins are made up of copper or aluminium. Of course, aluminium is the preferred material from cost and weight considerations.
2. Large ratio of perimeter to area of cross section is desirable; that means, thin, closely spaced fins are preferable. However, fins should not be too close as to impede the flow of fluid by convection.
3. Fins are justified when heat transfer coefficient h is small, i.e. generally on the gas side of a heat exchanger rather than on the liquid side. For example, the car radiator has fins on the outside of the tubes across which air flows.
4. Requirement that ε_f > 2, gives us the criterion:

$$\frac{k.P}{h.A_c} > 4 \dots \dots ..18$$

These two important parameters, namely, η_f and ε_f are related to each other as follows:

$$\varepsilon_f = \frac{Q_{fin}}{Q_{base}} = \frac{Q_{fin}}{h.A_c.(T_o - T_a)} = \frac{\eta_f.h.A_f.(T_o - T_a)}{h.A_c.(T_o - T_a)}$$

$$\varepsilon_f = \frac{A_f}{A_c}\eta_f \ldots\ldots 19$$